Written Exam for the M.Sc. in Economics Autumn 2013 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

Exam date: **10/1-2014**

3-hour open book exam.

Please note there are a total of **8** questions which should **all be replied to**. That is, **4** questions under *Question A*, and **4** under *Question B*.

Total numbers of pages (including this one): 5

Please also note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish. If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Question A:

Corresponding to some important event within the sample $t = 0, 1......, T$ at time $t = \tau$ say, we split the sample for a return series y_t in two parts: $t = 1, ..., \tau$ and $t = \tau+2, ..., T$ respectively, with initial values y_0 and $y_{\tau+1}$ for each sample part. Moreover, each part of the sample is modelled by an ARCH model.

We formalize this as the following split-type ARCH (spARCH) model, given by

$$y_t = v_t \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t. \tag{1}$$

Here z_t i.i.d. $\mathrm{N}(0,1)$, $\sigma_t^2 = 1 - \alpha + \alpha \varepsilon_{t-1}^2$ and $v_t^2 = \omega + \gamma \delta_t$, with $\delta_t = 1$ for $t > \tau$ and $\delta_t = 0$ otherwise. Moreover, for estimation we condition on the initial value y_0 in the first sample, and $y_{\tau+1}$ in the second sample.

Question A.1: Consider the process y_t defined above, but only for the second part of the sample $t = \tau + 2, ..., T$ and initial value $y_{\tau+1}$. Apply the drift criterion (include derivations) and show for which values of the parameters (ω, γ, α) the process $(y_t)_{t=\tau+1,...,T}$ is weakly mixing with a stationary solution with $E(y_t^2) < \infty$. Find $E(y_t^2)$ for this sample.

Question A.2: With $\theta = (\omega, \gamma, \alpha)$, show that the log-likelihood function $L_{T,1}(\theta)$ for the sample $t = 2, ..., \tau$ with y_1 fixed, is up to a constant given by,

$$L_{T,1} = -\sum_{t=2}^{\tau} \left(\log \sigma_{1t}^2 + y_t^2 / \sigma_{1t}^2 \right).$$

State in particular what σ_{1t}^2 is. Next, state the log-likelihood function $L_{T,2}(\theta)$ for the sample $t = \tau + 2, ..., T$, with $y_{\tau+1}$ fixed, and give the log-likelihood function $L_T(\theta)$ for the full sample.

Define $s_1^2 := \frac{1}{\tau+1} \sum_{t=0}^{\tau} y_t^2$ and $s_2^2 := \frac{1}{T-\tau} \sum_{t=\tau+1}^{T} y_t^2$. Explain how you can use these in a two-step estimation procedure where you first find $\hat{\omega}, \hat{\gamma}$, and then in the second step, $\hat{\alpha}$. Comment on whether you expect this to be as good an estimator of $\theta = (\omega, \gamma, \alpha)$ as the MLE of θ.

Question A.3: Show that the score for the second sample in terms of γ evaluated at the true value $\theta_0 = (\omega_0, \alpha_0, \gamma_0)$, is given by,

$$S_{T,2}(\gamma) := \left(\partial L_{T,2}(\theta) / \partial \gamma \right)\big|_{\theta=\theta_0} = \sum_{t=\tau+2}^{T} \left(1 - z_t^2\right)(1 - \alpha_0) / \left((\omega_0 + \gamma_0)(1 - \alpha_0) + \alpha_0 y_{t-1}^2 \right).$$

Show that $\left(1/\sqrt{T}\right) S_{T,2}(\gamma) \xrightarrow{D} N(0, 2\kappa)$, where

$$\kappa = E\left((1-\alpha_0) / \left((\omega_0 + \gamma_0)(1-\alpha_0) + \alpha_0 y_{t-1}^2\right)\right)^2 < \infty.$$

Question A.4. Consider Figure A. The top graph shows a return series x_t for $t = 1, ..., T$ with $T = 1000$. Now, rather than the spARCH above, we apply the spline-GARCH model as given by,

$$x_t = v_t^* \sigma_t z_t,$$

where $\log (v_t^*)^2 = c + \sum_{j=1}^{k} \gamma_j 1(t > \tau_j)$, where k denotes the number of knots, $1 < \tau_1 < \tau_2 ... < \tau_{k-1} < \tau_k < T$ and $1(t > \tau_j) = 1$ if $t > \tau_j$ and zero otherwise. Also $\sigma_t^2 = 1 - a + a\left(x_{t-1}/v_{t-1}^*\right)^2 + b\sigma_{t-1}^2$ and z_t are i.i.d.N$(0, 1)$.

The lower part in Figure A shows the estimated $\hat{h}_t := (\hat{v}_t^* \hat{\sigma}_t)^2$ together with the spline $\hat{v}_t^*$ for $k = 5$ knots. Comment on the return series x_t in terms of the previous spARCH model. Comment on the Spline-GARCH estimated $\hat{h}_t$ and $\hat{v}_t^{*2}$.

Table A provides some output from modeling x_t as a classic GARH(1,1) model (where $v_t^* = 1$). Comment on all the output in the table.

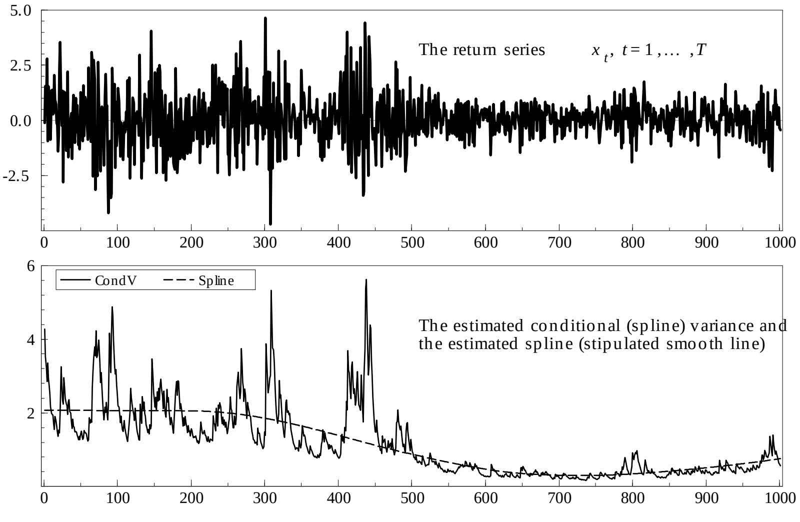

Figure A: Returns, x_t

Table A		
$\hat{a} = 0.11$	Normality test for standardized residuals:	p-value=0.02
$\hat{b} = 0.89$	ARCH test:	p-value=0.22

Question B:

Consider the threshold SV (TRSV) model as given by,

$$y_t = \sigma_t z_t \tag{B.1}$$

$$\log \sigma_t^2 = \rho \delta_t \log \sigma_{t-1}^2 + \xi_t, \tag{B.2}$$

where z_t and ξ_t are independent, with z_t i.i.d.N$(0,1)$ and ξ_t i.i.dN$\left(0, \sigma_\xi^2\right)$. Moreover, $\delta_t = 1\left(\left|\log \sigma_{t-1}^2\right| \leq \gamma\right)$, for some positive threshold parameter $\gamma > 0$. That is, $\delta_t = 1$ if $\left|\log \sigma_{t-1}^2\right| \leq \gamma$ and zero otherwise.

Question B.1: Show that $\log \sigma_t^2$ is weakly mixing and has a stationary representation for $\rho \in \mathbb{R}$ and $\gamma > 0$. In particular, explain how it can be the case that $\rho = 1$ is allowed in this case. Does this imply anything for joint evolution of y_t and σ_t^2?

Question B.2: Set $Y_t := \log y_t^2 - \mu$, where $\mu = E \log z_t^2$. Moreover, set $X_t := \log \sigma_t^2$. It follows that the system for Y_t and X_t can be written as,

$$Y_t = X_t + \varepsilon_t, \quad X_t = g\left(X_{t-1}\right) X_{t-1} + \xi_t.$$

Here ε_t is i.i.d.$(0, \sigma_\varepsilon^2)$. Moreover, $g\left(X_{t-1}\right) = \rho 1\left(|X_{t-1}| \leq \gamma\right)$.

Argue that ε_t is not Gaussian, but that ε_t is an iid sequence with mean zero and variance $\sigma_\varepsilon^2 = \pi^2/2$.

Observe that with $Y_{1:t} = (Y_1, ..., Y_t)$, $E\left(Y_t | Y_{1:t-1}\right) = X_{t|t-1} = E\left(X_t | Y_{1:t-1}\right)$. Hence one would be interested in computing $X_{t|t-1} = E\left(X_t | Y_{1:t-1}\right)$ as a function of $X_{t-1|t-1} = E\left(X_{t-1} | Y_{1:t-1}\right)$.

However, the *linear* Kalman filter would not work here. Explain why.

Question B.3: Provide an outline of how you would find the MLE of $\theta = \left(\rho, \gamma, \sigma_\xi^2\right)$ based on the particle filter.

In particular, relate your explanation to what happens below in the piece of code from ox.

```
DrawProposal(x_1)
{
decl x;
x = rho*x_1*(fabs(x_1)<=gamma) + sqrt(sigma)*rann(1,1);
return x;
}
```

Question B.4:

Figure B below shows a return series x_t, $t = 1, 2, ..., T$, simulated from the model in (B.1)-(B.2) with $\rho = 1$. Moreover, it shows two estimated conditional variance series: One series, $\hat{\sigma}^2_{t,GARCH}$, is found by a simple linear GARCH(1,1) estimation for x_t. The other series, $\hat{\sigma}^2_{t,MLE}$, is the comparable variance estimated by MLE using the particle filter.

Comment on the return series and the estimated conditional variances.

Explain how $\hat{\sigma}^2_{t,MLE}$ could be obtained by a particle filter. Explain, in particular, the difference between the predicted and the filtered volatility. Elaborate.

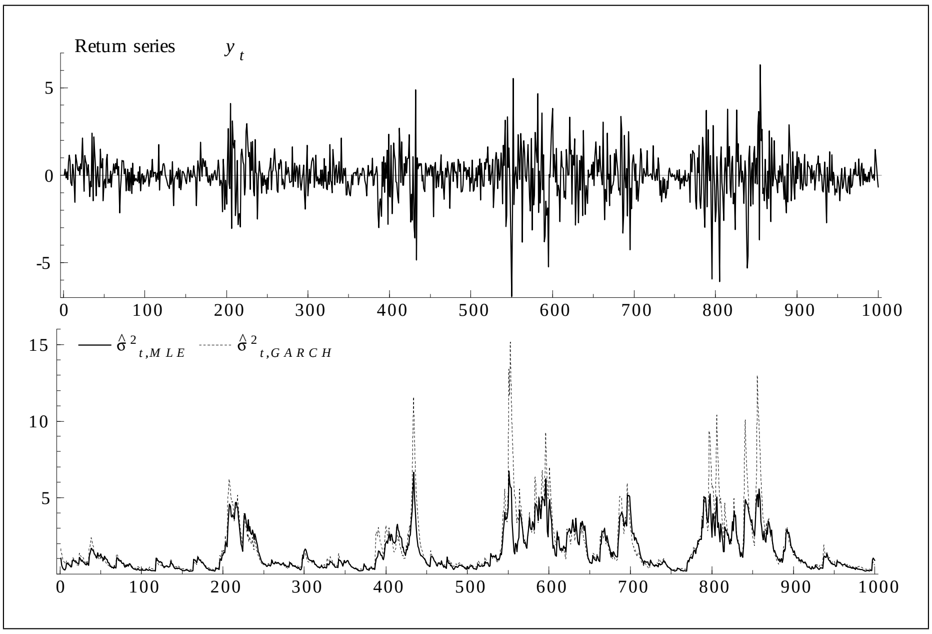

Figure B